Minimization Problem with Smooth Components

Yu. Nesterov
Presenter: Lei Tang

Department of CSE
Arizona State University

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Outline

- MiniMax problem
- Gradient Mapping for MiniMax problem ;
- The complexity of gradient and optimal method;
- Optimization with functional constraint (General constrained optimization problem)
- Constrained Minimization Problem

MiniMax Problem

- Objective function is composed with several components.
- The simplest problem of that type is *minimax* problem.
- We'll focus on smooth minimax problem:

$$\min_{x \in Q} \left[f(x) = \max_{1 \leq i \leq m} f_i(x) \right]$$

where $f_i \in \mathcal{S}^{1,1}_{\mu,L}(R^n), \quad i = 1, \cdots, m$ and Q is a closed convex set.

- $f(x)$: the max-type function composed by the components $f_i(x)$.

- In general, $f(x)$ is not differentiable.
- We use $f \in \mathcal{S}^{1,1}_{\mu,L}(R^n)$ to denote all the $f_i \in \mathcal{S}^{1,1}_{\mu,L}(R^n)$.

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Connection with General Minimization Problem

General Minimization Problem

$$\begin{aligned} \min \quad & f_0(x) & (1)\\ s.t. \quad & f_i(x) \leq 0, \quad i = 1, \cdots, m & (2)\\ & x \in Q & (3) \end{aligned}$$

parametric max-type function

$$f(t; x) = \max\{f_0(x) - t; f_i(x)\}$$

Will be showed later:

- the optimal value of $f_0(x)$ corresponds to the root t of $f(t; x) = 0$;
- minimax problem is used as a subroutine to solve (1);

Linear approximation

Linearization

max-type function $f(x) = \max_{1 \le i \le m} f_i(x)$

linearization of $f(x)$ $f(\bar{x}; x) = \max_{1 \le i \le m} [f_i(\bar{x}) + \langle f_i'(\bar{x}), x - \bar{x} \rangle]$

Essentially, linearization over each component.

Properties

- $f(\bar{x}; x) + \frac{\mu}{2}||x - \bar{x}||^2 \le f(x) \le f(\bar{x}; x) + \frac{L}{2}||x - \bar{x}||^2$;
- $x^* \in Q \Leftrightarrow f(x^*; x) \ge f(x^*; x^*) = f(x^*)$.
- $f(x) \ge f(x^*) + \frac{\mu}{2}||x - x^*||^2$
- the solution x^* exists and unique.

Lemma 2.3.1

$f(\bar{x};x) + \frac{\mu}{2}||x-\bar{x}||^2 \leq f(x) \leq f(\bar{x};x) + \frac{L}{2}||x-\bar{x}||^2$

- $f_i \in \mathcal{S}^{1,1}_{\mu,L}(R^n)$
- For strongly convex function, we have

$$\begin{aligned} f_i(x) &\geq f_i(\bar{x}) + \langle f_i'(\bar{x}, x-\bar{x})\rangle + \frac{\mu}{2}||x-\bar{x}||^2 \\ &= f(\bar{x};x) + \frac{\mu}{2}||x-\bar{x}||^2 \end{aligned}$$

Take the max on both sides: $f(x) \geq f(\bar{x};x) + \frac{\mu}{2}||x-\bar{x}||^2$

- For Lipshitz continuous function, it follows

$$\begin{aligned} f_i(x) &\leq f_i(\bar{x}) + \langle f_i'(\bar{x}, x-\bar{x})\rangle + \frac{L}{2}||x-\bar{x}||^2 \\ &= f(\bar{x};x) + \frac{L}{2}||x-\bar{x}||^2 \end{aligned}$$

- max operation keeps the property as smooth strongly convex function.

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Theorem 2.3.1: $x^* \in Q \Leftrightarrow f(x^*; x) \geq f(x^*; x^*) = f(x^*)$

$\Leftarrow$ As $f(x) \geq f(\bar{x}; x) + \frac{\mu}{2}||x - \bar{x}||^2$, we have

$$f(x) \geq f(x^*; x) + \frac{\mu}{2}||x - x^*||^2 \geq f(x^*; x^*) + 0 = f(x^*)$$

$\Rightarrow$ Prove by contradiction: if $f(x^*; x) < f(x^*)$, then for $1 \leq i \leq m$

$$f_i(x^*) + \langle f'(\bar{x}; x^*), x - x^* \rangle < f(x^*) = \max_{1 \leq i \leq m} f_i(x^*)$$

Define $\phi_i(\alpha) = f_i(x^* + \alpha(x - x^*)), \quad \alpha \in [0, 1]$
So either $\phi_i(0) \equiv f_i(x^*) < f(x^*)$ or

$$\phi_i(0) = f(x^*), \quad \phi_i'(0) = \langle f_i'(x^*), x - x^* \rangle < 0$$

So small enough α,

$$f_i(x^* + \alpha(x - x^*)) = \phi_i(\alpha) < f(x^*) \quad \forall 1 \leq i \leq m$$

contradiction!

Linearization achieves its minimum at x^*

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Linearization achieves its minimum at x^*

Corollary 2.3.1

$$f(x) \geq f(x^*) + \frac{\mu}{2}||x - x^*||^2$$

$$\begin{aligned} f(x) &\geq f(\bar{x}; x) + \frac{\mu}{2}||x - \bar{x}||^2 \\ &\geq f(x^*; x) + \frac{\mu}{2}||x - x^*||^2 \\ &\geq f(x^*; x^*) + \frac{\mu}{2}||x - x^*||^2 \\ &= f(x^*) + \frac{\mu}{2}||x - x^*||^2 \end{aligned}$$

So if x^* exists, it must be unique.

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Theorem 3.2

Let a max-type function $f(x) \in \mathcal{S}_\mu^1(R^n)$, $\mu > 0$, and Q be a closed convex set. Then the solution x^* exists and unique.

- Let $\bar{x} \in Q$, consider the set $\bar{Q} = \{x \in Q | f(x) \leq f(\bar{x})\}$.
- Transform to a problem as

$$\min\{f(x) | x \in \bar{Q}\}$$

- Need to show $\bar{Q}$ is bounded.

$$\begin{aligned} & f(\bar{x}) \geq f_i(x) \geq f_i(\bar{x}) + \langle f'(\bar{x}), x - \bar{x}\rangle + \frac{\mu}{2}||x - \bar{x}||^2 \\ \Longrightarrow \quad & \frac{\mu}{2}||x - \bar{x}||^2 \leq ||f'(\bar{x})|| \cdot ||x - \bar{x}|| + f(\bar{x}) - f_i(\bar{x}) \end{aligned}$$

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Quick Summary

MiniMax, though generally not smooth, share all the properties as minimizing smooth strongly convex functions over simple convex set.

Linearization

max-type function	$f(x) = \max_{1 \le i \le m} f_i(x)$
linearization of $f(x)$	$f(\bar{x}; x) = \max_{1 \le i \le m} [f_i(\bar{x}) + \langle f_i'(\bar{x}), x - \bar{x} \rangle]$

Essentially, linearization over each component.

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- $x^* \in Q \Leftrightarrow f(x^*; x) \ge f(x^*; x^*) = f(x^*)$.
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Road Map

- MiniMax problem
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As expected, share most of the properties as minimization over simple convex set.

Gradient Mapping

Similar as the case on minimization with convex set, we can define gradient mapping as follows:

$$f_\gamma(\bar{x}; x) = f(\bar{x}; x) + \frac{\gamma}{2}||x - \bar{x}||^2 \quad \text{(quadratic approximation)} \tag{4}$$

$$f^*(\bar{x}; \gamma) = \min_{x \in Q} f_\gamma(\bar{x}; x) \tag{5}$$

$$x_f(\bar{x}; \gamma) = \operatorname*{argmin}_{x \in Q} f_\gamma(\bar{x}; x) \tag{6}$$

$$g_f(\bar{x}; \gamma) = \gamma(\bar{x} - x_f(\bar{x}; \gamma)) \quad \text{(gradient mapping)} \tag{7}$$

The only difference is the linearization part $f(\bar{x}; x)$.

- When $m = 1$(only one component), the same as minimization over simple convex set;
- the linearization point $\bar{x}$ does not necessarily belong to Q;
- $f_\gamma(\bar{x}; x)$ is a max-type function composed with components:

$$f_i(\bar{x}) + \langle f_i'(\bar{x}), x - \bar{x} \rangle + \frac{\gamma}{2}||x - \bar{x}||^2 \in \mathcal{S}^{1,1}_{\gamma,\gamma}(R^n), \quad i = 1, \cdots, m \tag{8}$$

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f_\gamma(\bar{x}; x) &= f(\bar{x}; x) + \frac{\gamma}{2}||x - \bar{x}||^2 \qquad \text{(quadratic approximation)} && (4)\\
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Linearization and gradient mapping

$f(x)$ is bounded by the linearization (plus quadratic term), Could we somehow bound the linearization part with gradient mapping?

Theorem 2.3.3

Let $f \in \mathcal{S}^{1,1}_{\mu,L}(R^n)$, then for all $x \in Q$

$$f(\bar{x};x) \geq f^*(\bar{x};\gamma) + \langle g_f(\bar{x};\gamma), x - \bar{x}\rangle + \frac{1}{2\gamma}||g_f(\bar{x};\gamma)||^2 \quad (9)$$

$$\begin{aligned}
f(\bar{x};x) &= f_\gamma(\bar{x};x) - \frac{\gamma}{2}||x-\bar{x}||^2 && (10)\\
&\geq \underbrace{f_\gamma(\bar{x};x_f) + \frac{\gamma}{2}(||x-x_f||^2 - ||x-\bar{x}||^2)}_{f_\gamma(\bar{x};x)\in\mathcal{S}^{1,1}_{\gamma,\gamma}(R^n)} && (11)\\
&= f^*(\bar{x};\gamma) + \frac{\gamma}{2}\langle(\bar{x}-x_f, 2x - x_f - \bar{x}\rangle && (12)\\
&= f^*(\bar{x};\gamma) + \frac{\gamma}{2}\langle(\bar{x}-x_f, 2(x-\bar{x}) + (\bar{x}-x_f)\rangle && (13)\\
&= f^*(\bar{x};\gamma) + \langle g_f, x-\bar{x}\rangle + \frac{1}{2\gamma}||g_f||^2 && (14)
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\end{aligned}$$

Properties with respect to gradient mapping

Since $f(\bar{x}; x) \geq f^*(\bar{x}; \gamma) + \langle g_f(\bar{x}; \gamma), x - \bar{x} \rangle + \frac{1}{2\gamma} ||g_f(\bar{x}; \gamma)||^2$

Corollary 2.3.2 *Let $f \in \mathcal{S}^{1,1}_{\mu,L}(R^n)$ and $\gamma \geq L$. Then:*

1. For any $x \in Q$ and $\bar{x} \in R^n$ we have:

$$f(x) \geq f(x_f(\bar{x}; \gamma)) + \langle g_f(\bar{x}; \gamma), x - \bar{x} \rangle + \frac{1}{2\gamma} \| g_f(\bar{x}; \gamma) \|^2 + \frac{\mu}{2} \| x - \bar{x} \|^2 . \tag{2.3.7}$$

2. If $\bar{x} \in Q$ then

$$f(x_f(\bar{x}; \gamma)) \leq f(\bar{x}) - \frac{1}{2\gamma} \| g_f(\bar{x}; \gamma) \|^2, \tag{2.3.8}$$

3. For any $\bar{x} \in R^n$ we have:

$$\langle g_f(\bar{x}; \gamma), \bar{x} - x^* \rangle \geq \frac{1}{2\gamma} \| g_f(\bar{x}; \gamma) \|^2 + \frac{\mu}{2} \| x^* - \bar{x} \|^2 . \tag{2.3.9}$$

Proof:
Assumption $\gamma \geq L$ implies that $f^*(\bar{x}; \gamma) \geq f(x_f(\bar{x}; \gamma))$. Therefore (2.3.7) follows from (2.3.6) since

$$f(x) \geq f(\bar{x}; x) + \frac{\mu}{2} \| x - \bar{x} \|^2$$

for all $x \in R^n$ (see Lemma 2.3.1).

Using (2.3.7) with $x = \bar{x}$, we get (2.3.8), and using (2.3.7) with $x = x^*$, we get (2.3.9) since $f(x_f(\bar{x}; \gamma)) - f(x^*) \geq 0$. □

Variance with respect to γ

Lemma 2.3.2 *For any γ_1, $\gamma_2 > 0$ and $\bar{x} \in R^n$ we have:*

$$f^*(\bar{x};\gamma_2) \geq f^*(\bar{x};\gamma_1) + \frac{\gamma_2 - \gamma_1}{2\gamma_1\gamma_2} \parallel g_f(\bar{x};\gamma_1) \parallel^2 .$$

Proof:
Denote $x_i = x_f(\bar{x};\gamma_i)$, $g_i = g_f(\bar{x};\gamma_i)$, $i = 1, 2$. In view of (2.3.6), we have:

$$f(\bar{x};x) + \frac{\gamma_2}{2} \parallel x - \bar{x} \parallel^2 \geq f^*(\bar{x};\gamma_1) + \langle g_1, x - \bar{x}\rangle + \frac{1}{2\gamma_1} \parallel g_1 \parallel^2 + \frac{\gamma_2}{2} \parallel x - \bar{x} \parallel^2 \qquad (2.3.10)$$

for all $x \in Q$. In particular, for $x = x_2$ we obtain:

$$\begin{aligned} f^*(\bar{x};\gamma_2) = {} & f(\bar{x};x_2) + \tfrac{\gamma_2}{2} \parallel x_2 - \bar{x} \parallel^2 \\ \geq {} & f^*(\bar{x};\gamma_1) + \langle g_1, x_2 - \bar{x}\rangle + \tfrac{1}{2\gamma_1} \parallel g_1 \parallel^2 + \tfrac{\gamma_2}{2} \parallel x_2 - \bar{x} \parallel^2 \\ = {} & f^*(\bar{x};\gamma_1) + \tfrac{1}{2\gamma_1} \parallel g_1 \parallel^2 - \tfrac{1}{\gamma_2}\langle g_1, g_2\rangle + \tfrac{1}{2\gamma_2} \parallel g_2 \parallel^2 \\ \geq {} & f^*(\bar{x};\gamma_1) + \tfrac{1}{2\gamma_1} \parallel g_1 \parallel^2 - \tfrac{1}{2\gamma_2} \parallel g_1 \parallel^2 . \end{aligned}$$

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Gradient Method: Comparison

General Scheme for Gradient Method:

$$\begin{aligned} x_0 &\in Q, \\ x_{k+1} &= x_k - h g_f(x_k; L), \quad k = 0, \cdots \end{aligned}$$

On minimization over Minimax Problem (Same as over simple set)

If we choose $h \leq \frac{1}{L}$ in General Scheme for Gradient Method, then

$$\|x_k - x^*\|^2 \leq (1 - \mu h)^k \|x_0 - x^*\|^2. \tag{15}$$

If $h = \frac{1}{L}$

$$\|x_k - x^*\|^2 \leq \left(1 - \frac{\mu}{L}\right)^k \|x_0 - x^*\|^2 \tag{16}$$

the gradient method has the same rate of convergence as in the smooth case.

Let $r_k = \|x_k - x^*\|, g = g_f(x_k; L)$, (As $2\langle g, x_k - x*\rangle \geq \frac{1}{\gamma}\|g\|^2 + \mu\|x_k - x^*\|^2$)

$$\begin{aligned} r_{k+1}^2 &= \|x_k - x^* - h_{g_f}\|^2 \\ &= r_k^2 - 2h\langle g_f, x_k - x^*\rangle + h^2\|g_f\|^2 \\ &\leq (1 - h\mu)r_k^2 + h(h - \frac{1}{L})\|g_f\|^2 \leq (1 - \frac{\mu}{L})r_k^2. \end{aligned}$$

Minimization Method - Optimal Method

- **Step 1: define the estimate sequence** Assume that we have $x_0 \in Q$. Define

$$\phi_0(x) = \phi_0^* + \frac{\gamma_0}{2}\|x - v_0\|^2, \quad (17)$$

$$\phi_k(x) = (1-\alpha_k)\phi_k + \alpha_k\left[f(x_Q) + \langle g_Q, x - y_k\rangle + \frac{1}{2\gamma}\|g_Q\|^2 + \frac{\mu}{2}\|x - y_k\|^2\right], \quad (18)$$

where $x_Q = x_Q(y_k; L)$ and $g_Q = g_Q(y_k; L)$.

- **Step 2: rewrite the sequence** $\{\phi_k(x)\}$ For $k \geq 0$, we have

$$\phi_k(x) = \phi_k^* + \frac{\gamma_k}{2}\|x - v_k\|^2, \quad (19)$$

where the following recursive rules are defined for γ_k, v_k, and ϕ_k^* as

$$\gamma_{k+1} = (1-\alpha_k)\gamma_k + \alpha_k\mu, \quad (20)$$

$$v_{k+1} = \frac{1}{\gamma_{k+1}}[(1-\alpha_k)\gamma_k v_k + \alpha_k\mu y_k - \alpha_k g_Q], \quad (21)$$

$$\phi_{k+1}^* = (1-\alpha_k)\phi_k^* + \alpha_k f(x_Q) + \left(\frac{\alpha_k}{2L} - \frac{\alpha_k^2}{2\gamma_{k+1}}\right)\|g_Q\|^2 + \frac{\alpha_k(1-\alpha_k)\gamma_k}{\gamma_{k+1}}\left(\frac{\mu}{2}\|y_k - v_k\|^2 + \langle g_Q, v_k - y_k\rangle\right). \quad (22)$$

- **Step 3: ensure** $\phi_k^* \geq f(x_k)$ Using the inequality

$$f(x_k) \geq f(x_Q) + \langle g_Q, x_k - y_k \rangle + \frac{1}{2\gamma}\|g_Q\|^2 + \frac{\mu}{2}\|x_k - y_k\|^2, \quad (23)$$

we come to the following lower bound

$$\begin{aligned} \phi_{k+1}^* \quad &\geq \quad (1-\alpha_k)f(x_k) + \alpha_k f(x_Q) + \left(\frac{\alpha_k}{2L} - \frac{\alpha_k^2}{2\gamma_{k+1}}\right)\|g_Q\|^2 \\ &\quad + \frac{\alpha_k(1-\alpha_k)\gamma_k}{\gamma_{k+1}}\left(\frac{\mu}{2}\|y_k - v_k\|^2 + \langle g_Q, v_k - y_k \rangle\right) \\ &\geq \quad f(x_Q) + \left(\frac{1}{2L} - \frac{\alpha_k^2}{2\gamma_{k+1}}\right)\|g_Q\|^2 + (1-\alpha_k)\left\langle g_Q, \frac{\alpha_k\gamma_k}{\gamma_{k+1}}(v_k - y_k) + x_k - y_k \right\rangle. \end{aligned}$$

Therefore, we choose

$$\begin{aligned} x_{k+1} &= x_Q, \\ L\alpha_k^2 &= (1-\alpha_k)\gamma_k + \alpha_k\mu = \gamma_{k+1}, \\ y_k &= \frac{1}{\gamma_k + \alpha_k\mu}(\alpha_k\gamma_k v_k + \gamma_{k+1}x_k). \end{aligned}$$

Constant Step Scheme 3 for Simple Set

1. Choose $x_0 \in Q$ and $\alpha_0 \in (0,1)$. Set $y_0 = x_0, q = \mu/L$.
2. kth iteration ($k \geq 0$).
   - Compute $f(y_k)$ and $f'(y_k)$. Set
     $$x_{k+1} = x_Q. \tag{24}$$
   - Compute $\alpha_{k+1} \in (0,1)$ from the equation
     $$\alpha_{k+1}^2 = (1-\alpha_{k+1})\alpha_k^2 + q\alpha_{k+1},$$
     and set
     $$\beta_k = \frac{\alpha_k(1-\alpha)}{\alpha_k^2 + \alpha_{k+1}}, \quad y_{k+1} = x_{k+1} + \beta_k(x_{k+1} - x_k). \tag{25}$$

Note that only $\{x_k\}$ are feasible for Q, while $\{y_k\}$ can not be guaranteed to be feasible.

Completely identical to unconstrained case. The convergent rate is exactly the same as unconstrained case.

Convergence Rate

Theorem 2.3.5 *Let the max-type function* f *belong to* $\mathcal{S}^{1,1}_{\mu,L}(R^n)$. *If in (2.3.12) we take* $\alpha_0 \geq \sqrt{\frac{\mu}{L}}$, *then*

$$f(x_k) - f^* \leq \left[f(x_0) - f^* + \frac{\gamma_0}{2} \parallel x_0 - x^* \parallel^2 \right] \times \min \left\{ \left(1 - \sqrt{\frac{\mu}{L}}\right)^k, \frac{4L}{(2\sqrt{L} + k\sqrt{\gamma_0})^2} \right\},$$

where $\gamma_0 = \frac{\alpha_0(\alpha_0 L - \mu)}{1-\alpha_0}$.

$$\textbf{Scheme for } f \in \mathcal{S}^{1,1}_{\mu,L}(R^n) \qquad (2.3.13)$$

0. Choose $x_0 \in Q$. Set $y_0 = x_0$, $\beta = \frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}$.

1. kth iteration ($k \geq 0$). Compute $\{f_i(y_k)\}$ and $\{f_i'(y_k)\}$. Set

$$x_{k+1} = x_f(y_k; L), \qquad y_{k+1} = x_{k+1} + \beta(x_{k+1} - x_k).$$

Theorem 2.3.6 *For this scheme we have:*

$$f(x_k) - f^* \leq 2\left(1 - \sqrt{\frac{\mu}{L}}\right)^k (f(x_0) - f^*). \qquad (2.3.14)$$

Proof:
Scheme (2.3.13) corresponds to $\alpha_0 = \sqrt{\frac{\mu}{L}}$. Then $\gamma_0 = \mu$ and we get (2.3.14) since $f(x_0) \geq f^* + \frac{\mu}{2} \parallel x_0 - x^* \parallel^2$ in view of Corollary 2.3.1. □

Optimization with Functional Constraints

Problem 2.3.16

$$\min \quad f_0(x) \tag{26}$$
$$s.t. \quad f_i(x) \leq 0, \quad i = 1, \cdots, m \tag{27}$$
$$x \in Q \tag{28}$$

parametric max-type function

$$f(t; x) = \max\{f_0(x) - t; f_i(x)\}$$

$f(t; \cdot)$ are strongly convex in x. For any t, $x^*(t)$ exists and unique.

$$f^*(t) = \min_{x \in Q} f(t; x)$$

We'll try to get close to the solution using a process based on the approximate values of the function $f^* t(x)$ (aka. *sequential quadratic programming*)

Lemma 2.3.4

Lemma 2.3.4 *Let t^* be the optimal value of the problem (2.3.16). Then*

$$f^*(t) \leq 0 \quad \textit{for all } t \geq t^*,$$

$$f^*(t) > 0 \quad \textit{for all } t < t^*.$$

Proof:
Let x^* be the solution to (2.3.16). If $t \geq t^*$ then

$$f^*(t) \leq f(t; x^*) = \max\{f_0(x^*) - t; f_i(x^*)\} \leq \max\{t^* - t; f_i(x^*)\} \leq 0.$$

Suppose that $t < t^*$ and $f^*(t) \leq 0$. Then there exists $y \in Q$ such that

$$f_0(y) \leq t < t^*, \quad f_i(y) \leq 0, \; i = 1, \ldots, m.$$

Thus, t^* cannot be the optimal value of (2.3.16). □

Note that, as t increases, $f^*(t)$ decreases in a sense.
Hence, the smallest root of the function $f^*(t)$ corresponds to the optimal value of the problem of functional constraint.
Our goal is to form a process of finding the root.

Lemma 2.3.5 *For any $\Delta \geq 0$ we have:*

$$f^*(t) - \Delta \leq f^*(t+\Delta) \leq f^*(t).$$

Proof:
Indeed,

$$\begin{aligned} f^*(t+\Delta) &= \min_{x \in Q} \max_{1 \leq i \leq m} \{f_0(x) - t - \Delta; f_i(x)\} \\ &\leq \min_{x \in Q} \max_{1 \leq i \leq m} \{f_0(x) - t; f_i(x)\} = f^*(t), \\ f^*(t+\Delta) &= \min_{x \in Q} \max_{1 \leq i \leq m} \{f_0(x) - t; f_i(x) + \Delta\} - \Delta \\ &\geq \min_{x \in Q} \max_{1 \leq i \leq m} \{f_0(x) - t; f_i(x)\} - \Delta = f^*(t) - \Delta. \quad \square \end{aligned}$$

- Thus, $f^*(t)$ decreases in t and is Lipshitz continuous with the constant equal to 1.
- Keep in mind that here the property is satisfied for any max-type function like $f_\mu(\bar{x}; x)$ and $f_L(\bar{x}; x)$.

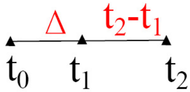
Lemma 2.3.6

For any $t_1 < t_2$ and $\Delta \geq 0$, we have

$$f^*(t_1 - \Delta) \geq f^*(t_1) + \Delta \frac{f^*(t_1) - f^*(t_2)}{t_2 - t_1} = f^*(t_1) - \Delta \frac{f^*(t_2) - f^*(t_1)}{t_2 - t_1} \quad (29)$$

Let $t_0 = t_1 - \Delta$, $\alpha = \frac{\Delta}{t_2 - t_0}$, then $t_1 = (1 - \alpha) t_0 + \alpha t_2$, so

$$(29) :\equiv f^*(t_1) \leq (1 - \alpha) f^*(t_0) + \alpha f^*(t_0)$$



Let $x_\alpha = (1 - \alpha) x^*(t_0) + \alpha x^*(t_2)$. We have:

$$f^*(t_1) \leq \max_{1 \leq i \leq m} \{f_0(x_\alpha) - t_1; f_i(x_\alpha)\}$$

$$\leq \max_{1 \leq i \leq m} \{(1 - \alpha)(f_0(x^*(t_0)) - t_0) + \alpha(f_0(x^*(t_2)) - t_2); (1 - \alpha) f_i(x^*(t_0)) + \alpha f_i(x^*(t_2))\}$$

$$\leq (1 - \alpha) \max_{1 \leq i \leq m} \{f_0(x^*(t_0)) - t_0; f_i(x^*(t_0))\} + \alpha \max_{1 \leq i \leq m} \{f_0(x^*(t_2)) - t_2; f_i(x^*(t_2))\}$$

$$= (1 - \alpha) f^*(t_0) + \alpha f^*(t_2),$$

Lemma 2.3.5

For any $\Delta > 0$, we have:

$$f^*(t) - \Delta \leq f^*(t + \Delta) \leq f^*(t)$$

Lemma 2.3.6

For any $t_1 < t_2$ and $\Delta \geq 0$, we have

$$f^*(t_1 - \Delta) \geq f^*(t_1) + \Delta \frac{f^*(t_1) - f^*(t_2)}{t_2 - t_1} = f^*(t_1) - \Delta \frac{f^*(t_2) - f^*(t_1)}{t_2 - t_1}$$

- Both Lemmas are valid for any parametric max-type functions.

Linearization and Gradient Mapping

Linearization of $f(t;x)$:

$$f(t;\bar{x};x) = \max_{1\le i\le m}\{f_0(x) + \langle f_0'(\bar{x}), x-\bar{x}\rangle - t;\quad f_i(x) + \langle f_i'(\bar{x}), x-\bar{x}\rangle\}$$

$$f_\gamma(t;\bar{x};x) = f(t;\bar{x};x) + \frac{\gamma}{2}\|x-\bar{x}\|^2 \quad (30)$$

$$f^*(t;\bar{x};\gamma) = \min_{x\in Q} f_\gamma(t;\bar{x};x) \quad (31)$$

$$x_f(t;\bar{x};\gamma) = \operatorname{argmin}_{x\in Q} f_\gamma(t;\bar{x};\gamma) \quad (32)$$

$$g_f(t;\bar{x};\gamma) = \gamma(\bar{x} - x_f(t;\bar{x};\gamma)) \quad (33)$$

g_f is the *constrained gradient mapping*; $\bar{x}$ is not necessarily in Q.

Bounds for the Linearization

$$f_\gamma(t;\bar{x};x) = f(t;\bar{x};x) + \frac{\gamma}{2}\|x-\bar{x}\|^2$$

- $f_\gamma(t;\bar{x};x)$ is itself a max-type function;
- $f_\gamma(t;\bar{x};x) \in \mathcal{S}^{1,1}_{\gamma,\gamma}(R^n)$. So for any t, the constrained gradient mapping is well defined;
- $f_\mu(t;\bar{x};x) \le f(t;x) \le f_L(t;\bar{x};x)$, as $f(t;x) \in \mathcal{S}^{1,1}_{\mu,L}(R^n)$; Hence

$$f^*_\mu(t;\bar{x};x) \le f^*(t) \le f^*_L(t;\bar{x};x)$$

- For any $\bar{x} \in R^n$, $\gamma > 0$, $\Delta \ge 0$ and $t_1 < t_2$, we have

$$f^*(t_1-\Delta;\bar{x};\gamma) \ge f^*(t_1;\bar{x};\gamma) + \frac{\Delta}{t_2-t_1}(f^*(t_1;\bar{x};\gamma) - f^*(t_2;\bar{x};\gamma))$$

- $f^*(t;\bar{x};\mu) \ge f^*(t;\bar{x};L) - \frac{L-\mu}{2\mu L}\|g_f(t;\bar{x};L)\|^2$

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Bounds for the Linearization

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- $f^*(t;\bar{x};\mu) \ge f^*(t;\bar{x};L) - \frac{L-\mu}{2\mu L}\|g_f(t;\bar{x};L)\|^2$

Root of $f^*(t; \bar{x}; \mu)$

- We are interested in finding the root of the function $f^*(t)$. We focus on the approximation of $f_\gamma(t; \bar{x}; \gamma)$.

$$t^*(\bar{x}; t) = root_t(f^*(t; \bar{x}; \mu))$$

- The root of the lower-bound quadratic approximation.
- Notice that the notation is a little confusing here $t^*(\bar{x}; t)$ actually depends on $\bar{x}$, not t.

Lemma 2.3.7 *Let $\bar{x} \in R^n$ and $\bar{t} < t^*$ are such that*

$$f^*(\bar{t}; \bar{x}; \mu) \geq (1 - \kappa) f^*(\bar{t}; \bar{x}; L)$$

for some $\kappa \in (0, 1)$. Then $\bar{t} < t^*(\bar{x}, \bar{t}) \leq t^*$. Moreover, for any $t < \bar{t}$ and $x \in R^n$ we have:

$$f^*(t; x; L) \geq 2(1 - \kappa) f^*(\bar{t}; \bar{x}; L) \sqrt{\frac{\bar{t} - t}{t^*(\bar{x}, \bar{t}) - \bar{t}}}.$$

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Lemma 2.3.7

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$$f^*(t; x; L) \geq 2(1-\kappa) f^*(\bar{t}; \bar{x}; L) \sqrt{\frac{\bar{t} - t}{t^*(\bar{x}, \bar{t}) - \bar{t}}}.$$

Proof:

Since $\bar{t} < t^*$, we have:

$$0 < f^*(\bar{t}) \leq f^*(\bar{t}; \bar{x}; L) \leq \frac{1}{1-\kappa} f^*(\bar{t}; \bar{x}; \mu).$$

Thus, $f^*(\bar{t}; \bar{x}; \mu) > 0$ and therefore $t^*(\bar{x}, \bar{t}) > \bar{t}$ since $f^*(t; \bar{x}; \mu)$ decreases in t.

Denote $\Delta = \bar{t} - t$, Then,

$$\begin{aligned} f^*(t;x;L) &\geq f^*(t) \geq f^*(t;\bar{x};\mu) && (34) \\ &\geq f^*(\bar{t};\bar{x};\mu) + \frac{\Delta}{t^*(\bar{x},t) - \bar{t}} \left[f^*(\bar{t};\bar{x};\mu) - \underbrace{f^*(t^*(\bar{x},t),\bar{x},\mu)}_{=0} \right] \\ &\geq (1-\kappa)\left(1 + \frac{\Delta}{t^*(\bar{x};t) - \bar{t}}\right) f^*(\bar{t};\bar{x};L) && (35) \\ &\geq (1-\kappa) 2\sqrt{\frac{\Delta}{t^*(\bar{x};t) - \bar{t}}} \; f^*(\bar{t};\bar{x};L) && (36) \\ &= 2(1-\kappa) f^*(\bar{t};\bar{x};L) \sqrt{\frac{\bar{t} - t}{t^*(\bar{x};t) - \bar{t}}} && (37) \end{aligned}$$

Constrained Minimization Scheme (2.3.22)

0. Choose $x_0 \in Q$ and $t_0 < t^*$. Choose $\kappa \in (0, \frac{1}{2})$ and the accuracy $\epsilon > 0$.

1. kth iteration ($k \geq 0$).

   a). Generate the sequence $\{x_{k,j}\}$ by the minimax method (2.3.13) as applied to the max-type function $f(t_k; x)$ with the starting point $x_{k,0} = x_k$. If

$$f^*(t_k; x_{k,j}; \mu) \geq (1-\kappa) f^*(t_k; x_{k,j}; L)$$

   then stop the internal process and set $j(k) = j$,

$$\begin{aligned} j^*(k) &= \arg \min_{0 \leq j \leq j(k)} f^*(t_k; x_{k,j}; L), \\ x_{k+1} &= x_f(t_k; x_{k,j^*(k)}; L). \end{aligned}$$

   Global Stop: Terminate the whole process if at some iteration of the internal scheme we have $f^*(t_k; x_{k,j}; L) \leq \epsilon$.

   b). Set $t_{k+1} = t^*(x_{k,j(k)}, t_k)$. □

Comment on the Scheme

Essentially two steps:

- Given t, find x until the lower bound $f(t;\bar{x};\mu)$ and the upper bound $f(t;\bar{x};L)$ of $f(t,\bar{x})$ is not too distant; Then pick the minimum one during the internal process;
- Given x, update t via finding the root of the lower bound;
  QCQP:

of the function

$$f^*(t;\bar{x};\mu) = \min_{x \in Q} f_\mu(t;\bar{x};x),$$

where $f_\mu(t;\bar{x};x)$ is a max-type function composed with the components

$$f_0(\bar{x}) + \langle f_0'(\bar{x}), x - \bar{x}\rangle + \tfrac{\mu}{2} \parallel x - \bar{x} \parallel^2 - t,$$

$$f_i(\bar{x}) + \langle f_i'(\bar{x}), x - \bar{x}\rangle + \tfrac{\mu}{2} \parallel x - \bar{x} \parallel^2, \; i = 1, \dots, m.$$

In view of Lemma 2.3.4, it is the optimal value of the following minimization problem:

$$\begin{aligned} &\min \, [f_0(\bar{x}) + \langle f_0'(\bar{x}), x - \bar{x}\rangle + \tfrac{\mu}{2} \parallel x - \bar{x} \parallel^2], \\ \text{s.t.} \quad & f_i(\bar{x}) + \langle f_i'(\bar{x}), x - \bar{x}\rangle + \tfrac{\mu}{2} \parallel x - \bar{x} \parallel^2 \le 0, \; i = 1, \dots, m, \\ & x \in Q. \end{aligned}$$

- The master process is continued until the upper bound function is close enough to 0 ($< \epsilon$)
- We start from a $t_0 < t^*$, and increases t gradually.

Following

- Here, we only focus on analytical complexity of this method.
- The total cost is of the order

$$\ln \frac{t_0 - t^*}{\epsilon} \sqrt{\frac{L}{\mu}} \ln \sqrt{\frac{L}{\mu}}$$

- This value differs from the lower bound for the unconstrained minimization problem by a factor of $\ln \frac{L}{\mu}$. (Not quite sure)
- Thus, the scheme is suboptimal for constrained optimization problems. But we cannot say more since the specific lower complexity bounds for constrained minimization are not known.
- We'll estimate the complexity of the master process;
- Then estimate the complexity for the internal process (given t, estimate an x);
- Finally, we get the total complexity.

Following

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- We'll estimate the complexity of the master process;
- Then estimate the complexity for the internal process (given t, estimate an x);
- Finally, we get the total complexity.

Lemma 2.3.8: complexity of master process

Lemma 2.3.8

$$f^*(t_k; x_{k+1}; L) \leq \frac{t^* - t_0}{1-\kappa}\left[\frac{1}{2(1-\kappa)}\right]^k$$

$$Let\ \beta = \frac{1}{2(1-\kappa)}(< 1\ as\ \kappa < 0.5) \qquad \delta_k = \frac{f^*(t_k; x_{k,j(k)}; L)}{\sqrt{t_{k+1} - t_k}}$$

Lemma 2.3.7

For $t < \bar{t} < t^*(\bar{x}, \bar{t}) \leq t^*$, we have

$$f^*(t; \bar{x}; L) \geq 2(1-\kappa) f^*(\bar{t}; \bar{x}; L)\sqrt{\frac{\bar{t} - t}{t^*(\bar{x}; t) - \bar{t}}} \tag{38}$$

Let $t = t_{k-1}, \bar{t} = t_k, t^*(\bar{x}; t) = t_{k+1}(As\ t_{k+1} = t^*(x_{k,j(k)}, t_k))$, we have

$$2(1-\kappa)\frac{f^*(t_k; x_{k,j(k)}; L)}{\sqrt{t_{k+1} - t_k}} \leq \frac{f^*(t_{k-1}; x_{k-1,j(k-1)}; L)}{\sqrt{t_k - t_{k-1}}} \implies \delta_k \leq \beta\delta_{k-1} \tag{39}$$

$$f^*(t_k; x_{k,j(k)}; L) = \delta_k\sqrt{t_{k+1} - t_k} \leq \beta^k \delta_0 \sqrt{t_{k+1} - t_k} \tag{40}$$

$$= \beta^k f^*(t_0; x_{x_0,j(0)}; L)\sqrt{\frac{t_{k+1} - t_k}{t_1 - t_0}} \tag{41}$$

Lemma 2.3.8: complexity of master process

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$$f^*(t_k; x_{k+1}; L) \leq \frac{t^* - t_0}{1-\kappa}\left[\frac{1}{2(1-\kappa)}\right]^k$$

$$Let\ \beta = \frac{1}{2(1-\kappa)} (< 1\ as\ \kappa < 0.5) \qquad \delta_k = \frac{f^*(t_k; x_{k,j(k)}; L)}{\sqrt{t_{k+1} - t_k}}$$

Lemma 2.3.5

For any $\Delta > 0$, we have:

$$f^*(t) - \Delta \leq f^*(t + \Delta) \leq f^*(t)$$

Let $t_1 = t_0 + \Delta$, we have $t_1 - t_0 \geq f^*(t_0; x_{0,j(0)}; \mu)$. So

$$f^*(t_k; x_{k,j(k)}; L) = \beta^k f^*(t_0; x_{x_0,j(0)}; L)\sqrt{\frac{t_{k+1} - t_k}{t_1 - t_0}} \quad (42)$$

$$\leq \beta^k f^*(t_0; x_{x_0,j(0)}; L)\sqrt{\frac{t_{k+1} - t_k}{f^*(t_0; x_{0,j(0)}; \mu)}} \quad (43)$$

$$\leq \frac{\beta^k}{1-\kappa}\sqrt{f^*(t_0; x_{0,j(0)}; \mu)(t_{k+1} - t_k)} \leq \frac{\beta^k}{1-\kappa}\sqrt{f^*(t_0)(t_0 - t^*)} \quad (44)$$

$$\leq \frac{t^* - t_0}{1-\kappa}\left[\frac{1}{2(1-\kappa)}\right]^k \quad (As\ \ f^*(t_0) \leq t^* - t_0) \quad (45)$$

Lemma 2.3.8: complexity of master process

Master Process

$$f^*(t_k; x_{k+1}; L) \leq \frac{t^* - t_0}{1-\kappa} \left[\frac{1}{2(1-\kappa)}\right]^k < \epsilon \Longrightarrow N(\epsilon) = \frac{1}{\ln[2(1-\kappa)]} \ln \frac{t^* - t_0}{(1-\kappa)\epsilon}$$

Complexity of Internal Process

Lemma 2.3.10 *For all k, $0 \le k \le N$, we have:*

$$j(k) \le 1 + \sqrt{\frac{L}{\mu}} \cdot \ln \frac{2(L-\mu)\Delta_k}{\kappa\mu\Delta_{k+1}}.$$

Corollary 2.3.3

$$\sum_{k=0}^{N} j(k) \le (N+1)\left[1 + \sqrt{\tfrac{L}{\mu}} \cdot \ln \tfrac{2(L-\mu)}{\kappa\mu}\right] + \sqrt{\tfrac{L}{\mu}} \cdot \ln \tfrac{\Delta_0}{\Delta_{N+1}}.$$

Lemma 2.3.11

$$j^* \le 1 + \sqrt{\frac{L}{\mu}} \cdot \ln \frac{2(L-\mu)\Delta_{N+1}}{\kappa\mu\epsilon}.$$

Corollary 2.3.4

$$j^* + \sum_{k=0}^{N} j(k) \le (N+2)\left[1 + \sqrt{\frac{L}{\mu}} \cdot \ln \frac{2(L-\mu)}{\kappa\mu}\right] + \sqrt{\frac{L}{\mu}} \cdot \ln \frac{\Delta_0}{\epsilon}.$$

As $N(\epsilon) = \frac{1}{\ln[2(1-\kappa)]} \ln \frac{t^* - t_0}{(1-\kappa)\epsilon}$, the total cost is:

$$\left[\tfrac{1}{\ln[2(1-\kappa)]} \ln \tfrac{t_0 - t^*}{(1-\kappa)\epsilon} + 2\right] \cdot \left[1 + \sqrt{\tfrac{L}{\mu}} \cdot \ln \tfrac{2(L-\mu)}{\kappa\mu}\right]$$
$$+ \sqrt{\tfrac{L}{\mu}} \cdot \ln\left(\tfrac{1}{\epsilon} \cdot \max_{1 \le i \le m}\{f_0(x_0) - t_0; f_i(x_0)\}\right). \quad (2.3.26)$$