A Convex Formulation for Learning Shared Structures from Multiple Tasks

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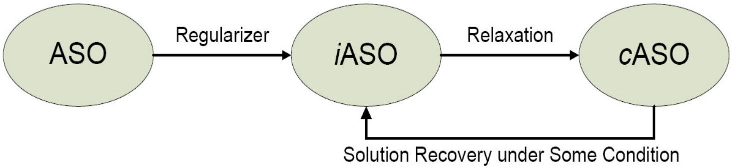
Our Motivation

- Multi-task learning aims to improve the generalization performance by learning from multiple related tasks.
    - Applied in the areas of machine learning, data mining, compute vision, and bioinformatics.
- Ando and Zhang (JMLR 05) propose the alternating structure optimization (ASO) algorithm to learn the predictive structure from multiple tasks.
- The ASO formulation is non-convex and its algorithm is not guaranteed to find a global optimum.

Main Contribution

- Propose an improved ASO formulation (*i*ASO) using a novel regularizer.
- Convert *i*ASO into a (relaxed) convex formulation, which is not scalable to large data sets.
- Propose a convex alternating structure optimization (*c*ASO) algorithm to efficiently find the globally optimal solution for the convex relaxation.
- Present a theoretical condition under which *c*ASO finds a globally optimal solution to *i*ASO.



Problem Setting

- Given m supervised learning tasks, where the ℓ-th tasks is associated with training data

$$\{(x_1^\ell, y_1^\ell), \cdots, (x_{n_\ell}^\ell, y_{n_\ell}^\ell)\} \subset \mathbb{R}^d \times \{-1, 1\}, \ \ell \in \mathbb{N}_m,$$

  and a linear predictor denoted as

$$f_\ell(x) = u_\ell^\mathsf{T} x, \ \ u_\ell \in \mathbb{R}^d.$$

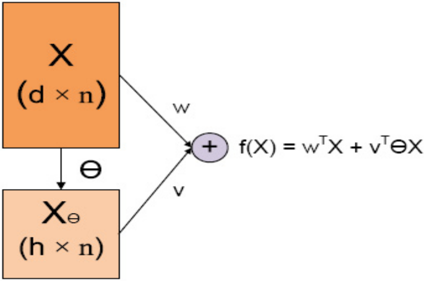
- Assume that the m learning tasks are related using some low dimensional feature space.

*i*ASO Formulation I

- We consider the linear predictor in the form of (Ando and Zhang, 2005)

$$f_\ell(x) = u_\ell^T x = w_\ell^T x + v_\ell^T \Theta x, \ \ \Theta\Theta^T = I. \tag{1}$$

  - Θ: the shared structure parameter
  - u_ℓ, w_ℓ, v_ℓ: the feature space weight vectors



*i*ASO Formulation II

- The proposed improved ASO formulation (*i*ASO) is given by:

$$(\mathbf{F_0}) \quad \min_{\{u_\ell, v_\ell\}, \Theta\Theta^T = I} \quad \sum_{\ell=1}^{m} \left(\frac{1}{n_\ell} \sum_{i=1}^{n_\ell} L(u_\ell^T x_i^\ell, y_i^\ell) + g_\ell(u_\ell, v_\ell, \Theta) \right),$$

  where L is the loss function, and $g_\ell(u_\ell, v_\ell, \Theta)$ is defined as:

$$g_\ell(u_\ell, v_\ell, \Theta) = \alpha \|u_\ell - \Theta^T v_\ell\|^2 + \beta \|u_\ell\|^2. \qquad (2)$$

  - $\|u_\ell - \Theta^T v_\ell\|^2$: control the task relatedness
  - $\|u_\ell\|^2$: control the complexity of the predictor functions

- If $\alpha = 0$, *i*ASO reduces to m independent SVMs. If $\beta = 0$, *i*ASO reduces to the ASO formulation.

Equivalent Reformulation I

- The objective function in *i*ASO:

$$\sum_{\ell=1}^{m}\left(\frac{1}{n_\ell}\sum_{i=1}^{n_\ell} L(u_\ell^{\mathsf{T}} x_i^\ell, y_i^\ell) + \alpha\|u_\ell - \Theta^{\mathsf{T}} v_\ell\|^2 + \beta\|u_\ell\|^2\right). \quad (3)$$

- **Reformulation 1:** The optimal $\{v_\ell\}$ to *i*ASO is given by $v_\ell = \Theta u_\ell$. By substitution, Eq. (3) can be written as

$$\sum_{\ell=1}^{m}\left(\frac{1}{n_\ell}\sum_{i=1}^{n_\ell} L(u_\ell^{\mathsf{T}} x_i^\ell, y_i^\ell) + \alpha u_\ell^{\top}\left((1+\frac{\beta}{\alpha})I - \Theta^{\top}\Theta\right) u_\ell\right). \quad (4)$$

Equivalent Reformulation I

- **Reformulation 2:** Let $\eta = \beta/\alpha$. Following the equality

$$(1+\eta)I - \Theta^\mathsf{T}\Theta = \eta(1+\eta)\left(\eta I + \Theta^\mathsf{T}\Theta\right)^{-1}, \tag{5}$$

Eq. (4) can be further rewritten as

$$\sum_{\ell=1}^{m}\left(\frac{1}{n_\ell}\sum_{i=1}^{n_\ell} L(u_\ell^\mathsf{T} x_i^\ell, y_i^\ell) + \alpha\eta(1+\eta)u_\ell^\top \left(\eta I + \Theta^\mathsf{T}\Theta\right)^{-1} u_\ell\right). \tag{6}$$

Equivalent Reformulation III

- **Reformulation 3:** Let $U = [u_1, \cdots, u_m]$. By substituting the matrices product $\Theta^\top\Theta$ using a matrix M, *i*ASO can be reformulated as

$$(\mathbf{F_1}) \quad \min_{U,M} \quad \sum_{\ell=1}^{m} \left(\frac{1}{n_\ell} \sum_{i=1}^{n_\ell} L(u_\ell^\mathsf{T} x_i^\ell, y_i^\ell) \right) + G_1(U, M)$$

$$\text{subject to} \quad M \in \left\{ M_e \mid M_e = \Theta^\mathsf{T}\Theta,\ \Theta\Theta^\mathsf{T} = I,\ \Theta \in \mathbb{R}^{h \times d} \right\},$$

where $G_1(U, M)$ is defined as

$$G_1(U, M) = \alpha\ \eta\ (1 + \eta)\ \mathrm{tr}\left(U^\mathsf{T} (\eta I + M)^{-1} U \right). \qquad (7)$$

Convex Relaxation I

- The convex hull of the set

$$\mathcal{M}_e = \left\{ M_e \,|\, M_e = \Theta^\mathsf{T}\Theta,\ \Theta\Theta^\mathsf{T} = I,\ \Theta \in \mathbb{R}^{h\times d} \right\} \tag{8}$$

can be precisely expressed as the convex set

$$\mathcal{M}_c = \left\{ M_c \,|\, \mathrm{tr}(M_c) = h,\ M_c \preceq I,\ M_c \in \mathbb{S}_+^d \right\}. \tag{9}$$

- Since $\mathcal{M}_c$ consists of all convex combinations of the elements in $\mathcal{M}_e$, $\mathcal{M}_c$ is the smallest convex set that contains $\mathcal{M}_e$.

Convex Relaxation II

- We convert the non-convex problem F_1 into a convex formulation as

$$
\begin{aligned}
(\mathbf{F_2}) \quad & \min_{U,M} \sum_{\ell=1}^{m} \left(\frac{1}{n_\ell} \sum_{i=1}^{n_\ell} L(u_\ell^{\mathrm{T}} x_i^\ell, y_i^\ell) \right) + G_2(U, M) \\
\text{subject to} \quad & \mathrm{tr}(M) = h, \;\; M \preceq I, \;\; M \in \mathbb{S}_+^d, \qquad (10)
\end{aligned}
$$

where

$$
G_2(U, M) = \alpha \, \eta \, (1 + \eta) \, \mathrm{tr} \left(U^{\mathrm{T}} \left(\eta I + M \right)^{-1} U \right). \qquad (11)
$$

SDP Formulation

- Following the Schur complement Lemma, we rewrite F_2 as

$$(\mathbf{F_3}) \quad \min_{U,M,\{t_\ell\}} \quad \sum_{\ell=1}^{m} \left(\frac{1}{n_\ell} \sum_{i=1}^{n_\ell} L(u_\ell^{\mathsf{T}} x_i^\ell, y_i^\ell) \right) + \alpha\eta(1+\eta) \sum_{\ell=1}^{m} t_\ell$$

$$\text{subject to} \quad \begin{pmatrix} \eta I + M & u_\ell \\ u_\ell^{\mathsf{T}} & t_\ell \end{pmatrix} \succeq 0, \ \forall \ell \in \mathbb{N}_m,$$

$$\mathrm{tr}(M) = h, \ M \preceq I, \ M \in \mathbb{S}_+^d. \tag{12}$$

- If L is the hinge loss function, F_3 is a SDP, which is not scalable to high-dimensional data.

Convex ASO Algorithm I

- We propose a convex alternating structure optimization (cASO) algorithm to efficiently solve F_2, that is, recycling between the following two steps:
  - Step 1: Given M, optimize U
  - Step 2: Given U, optimize M
- We can show that cASO finds the globally optimal solution to F_2 (Argyriou et al., 2007; Argyriou et al., 2008).

Convex ASO Algorithm II

- Given M, U can optimized via the problem:

$$\min_{U} \quad \sum_{\ell=1}^{m} \left(\frac{1}{n_\ell} \sum_{i=1}^{n_\ell} L(u_\ell^\mathrm{T} x_i^\ell, y_i^\ell) + \hat{g}(u_\ell) \right), \qquad (13)$$

where $\hat{g}(u_\ell)$ is given by

$$\hat{g}(u_\ell) = \alpha\, \eta\, (1+\eta)\, \mathrm{tr} \left(u_\ell^\mathrm{T} \left(\eta I + M\right)^{-1} u_\ell \right). \qquad (14)$$

- If L is the hinge loss, the problem in Eq. (13) decouples into m independent quadratic programs (QP).

Convex ASO Algorithm III

- Given U, M can be optimized via the problem:

$$\begin{aligned} \min_{M} \quad & \operatorname{tr}\left(U^{\mathrm{T}} (\eta I + M)^{-1} U\right) \\ \text{subject to} \quad & \operatorname{tr}(M) = h, M \preceq I, M \in \mathbb{S}_{+}^{d}. \end{aligned} \tag{15}$$

- Although Eq. (15) can be recast into an SDP, we propose an efficient approach to find its optimal solution.

Efficient Computation of M

- Given $U \in \mathbb{R}^{d \times m}$, the optimal M to Eq. (15) has an analytic form as

$$M = P_1 \Gamma^* P_1^\top, \ \Gamma^* = \mathrm{diag}\left(\gamma_1^*, \cdots, \gamma_q^*\right). \tag{16}$$

  - Step 1: Compute P_1 via the SVD of $U = P_1 \Sigma P_2^\top$, where

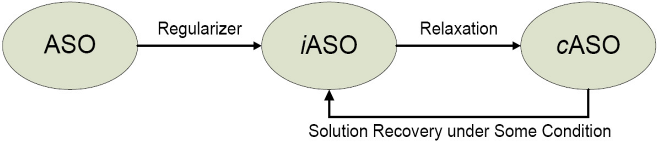
$$P_1 \in \mathbb{R}^{d \times d}, \Sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_q) \in \mathbb{R}^{d \times m}, \mathrm{rank}(U) = q.$$

  - Step 2: Compute $\{\gamma_i^*\}_{i=1}^q$ via solving:

$$\begin{aligned} \min_{\{\gamma_i\}_{i=1}^q} \quad & \textstyle\sum_{i=1}^q \frac{\sigma_i^2}{\eta + \gamma_i} \\ \text{subject to} \quad & \textstyle\sum_{i=1}^q \gamma_i = h, \ 0 \le \gamma_i \le 1, \ \forall i \in \mathbb{N}_q. \end{aligned} \tag{17}$$

Computation of an Optimal Solution to *i*ASO

- We show under a theoretical condition a globally optimal solution to *i*ASO (F_1) can be obtained from *c*ASO (F_2).
  - Let (U^*, M^*) be the optimal solution to F_2.
  - Let $P_1 \in \mathbb{R}^{d \times d}$ and $\{\sigma_i\}_{i=1}^q$ be the left singular vectors and the non-zero singular values of U^*, respectively.
  - If $\sigma_h / \sigma_{h+1} \geq 1 + 1/\eta$, the optimal solution to F_1 is given by (U^*, Θ^*), where Θ^* consist of the first h column of P_1.



Experimental Study I

Table: Performance comparison of competing algorithms.

Data Set (n, d, m)		Recreation (12797, 25095, 18)	Science (6345, 24002, 22)	Social (11914, 32492, 21)	Society (14507, 29189, 21)
Macro F1	SVM	43.01 ± 1.44	41.80 ± 1.45	35.87 ± 0.79	30.68 ± 0.94
	ASO	43.63 ± 1.29	39.26 ± 0.82	35.29 ± 0.67	29.42 ± 0.30
	*c*ASO	$\mathbf{47.12 \pm 0.73}$	$\mathbf{45.46 \pm 0.50}$	$\mathbf{39.30 \pm 1.28}$	$\mathbf{34.84 \pm 1.05}$
	*c*MTFL	46.13 ± 0.58	42.52 ± 0.59	38.94 ± 1.88	33.79 ± 1.43
Micro F1	SVM	49.15 ± 2.32	49.27 ± 4.64	63.05 ± 2.45	40.07 ± 3.42
	ASO	50.68 ± 0.18	49.05 ± 0.57	62.77 ± 3.59	46.13 ± 2.33
	*c*ASO	$\mathbf{53.34 \pm 0.90}$	$\mathbf{53.32 \pm 0.45}$	$\mathbf{66.04 \pm 0.62}$	$\mathbf{49.27 \pm 0.55}$
	*c*MTFL	52.52 ± 0.92	50.60 ± 0.76	65.60 ± 0.63	46.46 ± 0.87

Key Observation:

- *c*ASO outperforms or perform competitively with other competing algorithms.

Experimental Study II

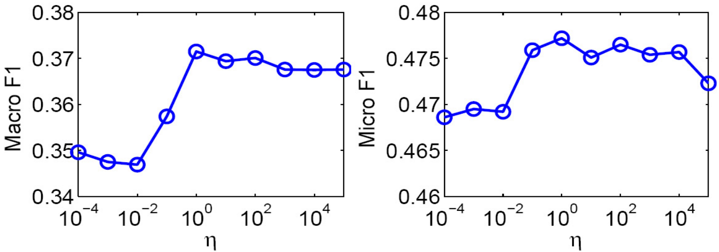


Figure 2. Sensitivity study of η on Micro/Micro F1.

Key Observation:

- A small η leads to lower F1, while $\eta \approx 1$ leads to the highest F1.
- *c*ASO requires more computation time for convergence using a small η, while less computation time is required for a large η.

Experimental Study III

Table: Comparison of the optimal objective values of F_0 and F_2 with different choices of η.

η	1000	100	10	1	0.1	0.01	0.001
$1 + 1/\eta$	1.001	1.01	1.1	2	11	101	1001
σ_h/σ_{h+1}	1.23	1.25	1.34	1.75	3.07	13.79	89.49
OBJ_{F_0}	52.78	52.65	51.37	40.73	22.15	5.95	0.69
OBJ_{F_2}	52.78	52.65	51.37	40.71	20.73	4.11	0.41

Key Observation:

- We can observe that when $\eta \in \{1000, 100, 10\}$, the condition $\sigma_h/\sigma_{h+1} > 1 + 1/\eta$ is satisfied and hence $\text{OBJ}_{F_0} = \text{OBJ}_{F_2}$; otherwise, we observe $\text{OBJ}_{F_0} > \text{OBJ}_{F_2}$.

Conclusion and Future Work

- Present *i*ASO for learning a shared feature representation from multiple related tasks.
- Convert *i*ASO into a relaxed convex formulation, and then develop the *c*ASO algorithm to compute its globally optimal solution efficiently.
- Present a theoretical condition, under which *c*ASO can find a globally optimal solution to *i*ASO.
- Plan to compare the *i*ASO formulation with the multi-task learning formulation using the trace-norm regularization.